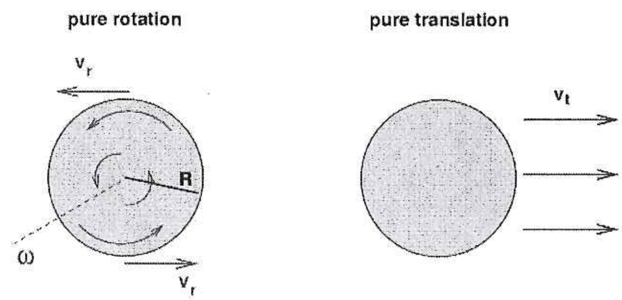
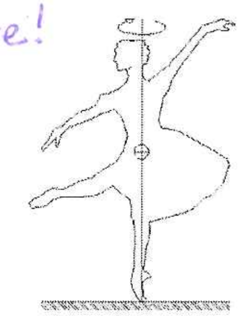
Chapter 8: Rotational Motion



We now shift our focus from translational motion (straight line motion of an objects center of mass) to rotational motion.

Rotational Motion – all points in a **rigid body** move in circles about the objects center, called the **axis of rotation**.

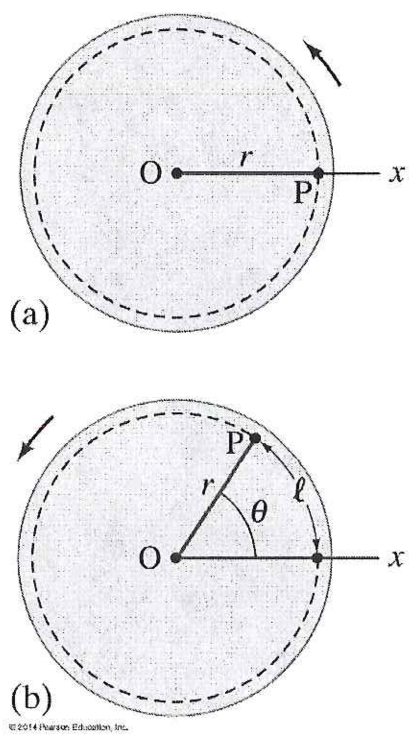
example: frisbees rotate and translate!



Rigid Body – a body with a definite shape so the particles composing it stay in fixed positions relative to one another.

Axis of Rotation – the imaginary line that passes through an objects' center of mass of which the object rotates about

Section 8-1 – Angular Quantities



Angular Displacement

Consider a rotating wheel rotating counter clockwise as shown in the diagram. It rotates about its center, point O, from its starting position (a) to its final position (b) in some amount of time, Δt.

Let's now look specifically at the motion of point P. The displacement of point P depends on how far it is from the axis of rotation, *r*. But, *every* point on the line between O and P travel through the ***same*** angle. We call this angle the **angular displacement** and can be defined using the arc length and *r*.

some move faster, but same angular displacement.

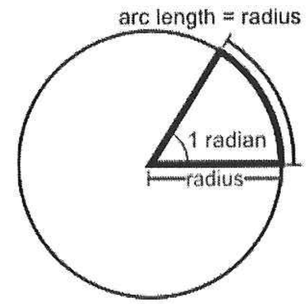
$$\Delta\theta \equiv \frac{\text{ArcLength}}{\text{distance from axis of rotation}} = \frac{\Delta\ell}{r}$$

Units for angular displacement:

θ is measured in **radians** because the mathematics of circular motion are much simpler if we use radians instead of degrees.

Radian (rad) – the angle *subtended* (www.mathopenref.com/subtend.html) by an arc whose length is equal to the radius.

How many degrees in 1 radian?

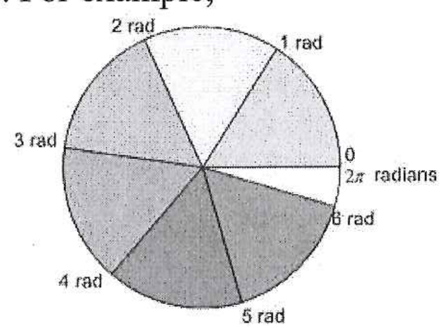


Start by imagining the point at a radius r traveling in a full circle. This is obviously an angle of 360°. The arc length, ℓ, must be the circumference of the circle, $2\pi r$. To get the angle in radians, we use:

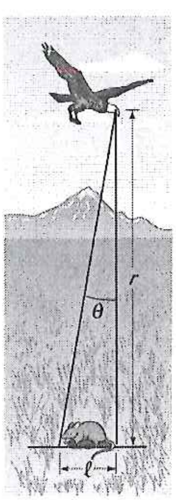
$$\Delta\theta = \frac{\Delta\ell}{r} = \frac{2\pi r}{r} = 2\pi \text{ radians}$$

So 360° is 2π rads. You can use this as your conversion factor. For example,

$$1 \text{ rad} \cdot \frac{360°}{2\pi \text{ rad}} \approx 57.3°$$



When you see the units of radians, remember that they just refer to angles.

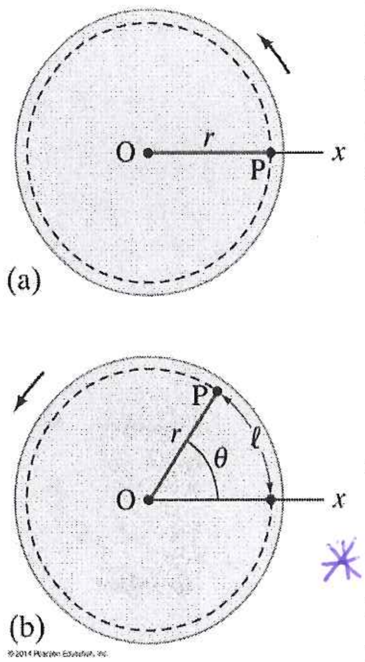


Ex 8.1 – A particular bird's eye can just distinguish objects that subtend an angle no smaller than about 3×10^{-4} rad. (a) How many degrees is this? (b) How small an object can the bird just distinguish when flying at a height of 100m?

a. $3\times10^{-4} \text{ rad} \times \frac{360°}{2\pi \text{ rad}} = \boxed{.0172°}$

b. $s = r\theta$ where $s \approx \ell$ for small θ and θ in radians

$s = 100(3\times10^{-4}) = .03\text{m} = \boxed{3\text{cm}}$



<u>Average angular velocity</u>

Back to our rotating wheel. We know that the wheel rotates through an **angular displacement**, θ, in a given amount of time. The point P travels an arc length, ℓ . Analogous to average velocity being displacement over time, **average angular velocity** (denoted by the Greek lowercase letter omega) is defined as angular displacement over time:

$$\boxed{\omega = \frac{\Delta\theta}{\Delta t}} \quad \text{(average angular velocity)}$$

The units are radians per second (rad/s).

*It is important to note that every point in a rigid body rotates with the *same angular velocity*, although they do move at different *linear velocities*.

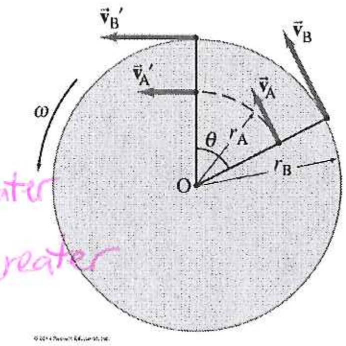
<u>Angular Velocity</u>

Remember that $\Delta\theta = \dfrac{\Delta\ell}{r}$ so $\Delta\ell = r\Delta\theta$. For small time intervals, the linear speed of any particle on the rigid body is $\boxed{v = \dfrac{\Delta\ell}{\Delta t}}$. This means:

$$v = \frac{\Delta\ell}{\Delta t} = \frac{r\Delta\theta}{\Delta t} = r\left(\frac{\Delta\theta}{\Delta t}\right) = \mathrm{r}\,\omega$$

$$\boxed{v = \omega\, r} \quad \text{(linear velocity; units are m/s)}$$

$$\omega = \frac{v}{r} \quad \text{(angular velocity; units are rad/s)}$$



Ex 8.2 – A rotating carousel has one child sitting on a horse near the outer edge and another child seated on a lion halfway out from the center. (a) Which child has the greater linear speed? (b) Which child has the greater angular speed?

a. $V = \frac{\text{distance}}{\text{time}}$ ⇒ child on edge travels a greater distance in same time and ∴ has a greater linear speed.

b. $\omega = \frac{\text{angular displacement}}{\text{time}}$ ⇒ they both do one full rotation (360°) in the same time and ∴ they have the same angular speed.

Average Angular Acceleration

Recall that when we studied circular motion there were two accelerations, centripetal and tangential. The ***angular acceleration*** (denoted by the lowercase Greek letter alpha) is analogous to tangential acceleration. It tells us how fast something is increasing or decreasing its rotational velocity. The ***average angular acceleration*** is defined as:

$$\boxed{\alpha \equiv \frac{\Delta\omega}{\Delta t}} \quad \text{(average angular acceleration; units are rad/s}^2\text{)}$$

If the ***angular acceleration*** is constant for a time interval, we can find it using

$$\alpha = \frac{\omega_f - \omega_i}{\Delta t}$$

Relating Angular Acceleration to Linear Acceleration

Recall that the linear speed is $v = \omega\, r$. Since the tangential acceleration is

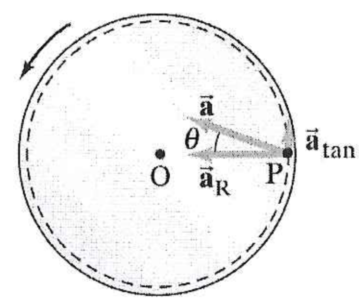
$$a_{\tan} = \frac{\Delta v}{\Delta t} = \frac{\Delta(\omega r)}{\Delta t} = (\frac{\Delta\omega}{\Delta t})r = \alpha r$$

$$\boxed{a_{\tan} = \alpha r} \quad \text{(tangential acceleration; m/s}^2\text{)}$$

$$\alpha = \frac{a_{\tan}}{r} \quad \text{(angular acceleration; rad/s}^2\text{)}$$

Total acceleration

We have related angular acceleration, but don't forget about the centripetal (radial) acceleration that points inward. Recall that:



(b)

$$a_c = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r$$

So the total acceleration is:

$$\vec{a}_{total} = \vec{\alpha}\, r + \vec{\omega}^2\, r$$

vector sum!

One interesting thing we can get from this equation; the centripetal acceleration depends linearly on *r*, the distance from the axis of rotation. This means that the further you are from the axis of rotation the greater centripetal acceleration and centripetal force you feel! Yay Physics!

Relating angular velocity to frequency of rotation

We can relate the angular velocity, ω, to the frequency of rotation, *f*, where by **frequency** we mean the number of complete revolutions (rev) per second. Since 1 revolution corresponds to an angular displacement of 2π radians, we have:

$$\omega = 2\pi f$$

Ex 8.3 – (a) What is the linear speed of a child seated 1.2m from the center of a steadily rotating merry-go-round that makes one complete revolution in 4.0s? (b) What is her acceleration?

$r = 1.2m$

$T = 4sec$

$f = .25Hz$

a. $v = \frac{2\pi r}{T} = \frac{2\pi(1.2m)}{4sec} = \boxed{1.88 m/s}$

(or $v = \omega r \Rightarrow v = 2\pi f r = 1.88 m/s$)

b. Since steadily rotating, $\alpha = 0\, rad/s^2$

$\therefore a_{total} = \omega^2 r$

$= \left(\frac{v}{r}\right)^2 r$

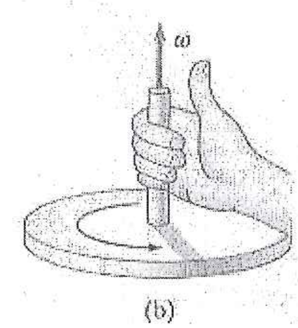
$= \frac{v^2}{r} = \frac{(1.88 m/s)^2}{1.2m}$

$\boxed{a_{total} = 2.96 m/s^2}$

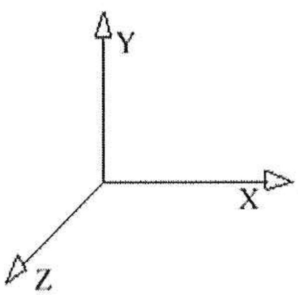
Direction for Angular Vectors

Consider a rotating wheel. Any linear direction in the plane of the wheel does not uniquely define the rotation of the wheel because of symmetry. So we define direction to be pointing *perpendicular* to the plane or rotation, and *parallel* to the axis of rotation.

The convention we use is called the *Right-Hand Rule.* You curl the fingers of your right hand in the direction of the rotation, and thumb will point in the direction of $\vec{\omega}$. This convention holds true for all angular vectors.



Right hand rule

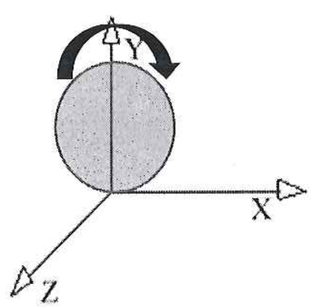


What should we call this direction? We will use the positive and negative z direction. If the x and y plane is oriented as usual, positive z points out of the page, and negative z points into the page.

Thus, we call an object rotating **counter-clockwise** to be in the **positive z** and an object rotating **clockwise** to be in the **negative z** direction.

<u>Example</u>: A wheel is rotating clockwise and its angular velocity is decreasing.

"righty tighty, lefty loosey!"



Since it is rotating clockwise, its angular velocity is in the negative z direction.
Since it is slowing down, its angular acceleration is in the positive z direction.

Equations from 8-1:

$\Delta\theta = \frac{\Delta \ell}{r}$ (angular displacement; rad)

$\omega = \frac{\Delta\theta}{\Delta t}$ $\omega = \frac{v}{r}$ $\boxed{\omega = 2\pi f}$ (angular velocity; rad/s)

$\alpha = \frac{\Delta\omega}{\Delta t}$ $\alpha = \frac{a_{\tan}}{r}$ (angular acceleration; rad/s^2)

8–1 Problems

1. A bicycle odometer (which counts revolutions and is calibrated to report distance traveled) is attached near the wheel axle and is calibrated for 27-inch wheels. What happens if you use it on a bicycle with 24-inch wheels?

For each revolution, it will think you went $2\pi(27)$ meters, but you only went $2\pi(24)$ meters so it will tell you a distance greater than what you actually traveled.

2. Express the following angles in radians: (*a*) 45.0°, (*b*) 60.0°, (*c*) 90.0°, (*d*) 360.0°, and (*e*) 445°. Give as numerical values and as fractions of π.

a. $45° \times \frac{2\pi \text{ rad}}{360°} = \boxed{\frac{\pi}{4} \text{ rad}}$ $\boxed{.785 \text{ rad}}$

b. $60° \times \frac{2\pi \text{ rad}}{360°} = \boxed{\frac{\pi}{3} \text{ rad}}$ $\boxed{1.047 \text{ rad}}$

c. $90° \times \frac{2\pi \text{ rad}}{360°} = \boxed{\frac{\pi}{2} \text{ rad}}$ $\boxed{1.57 \text{ rad}}$

d. $360° \times \frac{2\pi \text{ rad}}{360°} = \boxed{2\pi \text{ rad}}$ $\boxed{6.28 \text{ rad}}$

e. $445° \times \frac{2\pi \text{ rad}}{360°} = \boxed{\frac{89\pi}{36} \text{ rad}}$ $\boxed{7.767 \text{ rad}}$

3. (*a*) A grinding wheel 0.35 m in diameter rotates at 2200 rpm. Calculate its angular velocity in rad/s. (*b*) What is the linear speed and acceleration of a point on the edge of the grinding wheel?

a. $r = .35m$

$$f = 2200 \frac{rot}{min} \times \frac{2\pi \, rad}{1 \, rot} \times \frac{1 \, min}{60 \, sec} = \boxed{230.38 \, rad/sec} = \omega$$

b. $v = \omega r = 230.38(.35) = \boxed{80.63 \, m/s}$

$$a_{Rad} = \frac{v^2}{r} = \omega^2 r = \frac{80.63^2}{.35} \rightarrow \boxed{18576.79 \, m/s^2}$$

4. The blades in a blender rotate at a rate of 6500 rpm. When the motor is turned off during operation, the blades slow to rest in 4.0 s. What is the angular acceleration as the blades slow down?

$$\omega_i = 6500 \frac{rev}{min} \times \frac{2\pi \, rad}{1 \, rev} \times \frac{1 \, min}{60 \, sec} = 680.678 \, rad/sec$$

$t = 4sec$

$\omega_f = 0 \, rad/s$

$$\alpha = \frac{\Delta\omega}{t} = \frac{680.678 \, rad/sec}{4 \, sec}$$

$$\therefore \boxed{\alpha = 170.17 \, rad/s^2}$$

5. A child rolls a ball on a level floor 3.5 m to another child. If the ball makes 12.0 revolutions, what is its diameter?

$\Delta l = 3.5m$

$$\Delta\theta = 12 rev \times \frac{2\pi \, rad}{1 \, rev} = 24\pi \, rad$$

$$\Delta\theta = \frac{\Delta l}{r} \Rightarrow r = \frac{\Delta l}{\Delta\theta} = \frac{3.5m}{24\pi \, rad} = .046m$$

$$d = 2r \Rightarrow \boxed{d = .0928m}$$

Section 8-2 – Kinematic Equations for Uniformly Accelerated Rotational Motion

Recall the kinematic equations from chapter 2. They were derived assuming *constant linear acceleration*. Because the angular quantities are directly related to the linear quantities, we can derive four analogous equations for when there is *constant angular acceleration*.

Linear	Angular
$v_f = v_i + at$	$\omega_f = \omega_i + \alpha t$
$\Delta x = v_i t + \frac{1}{2}at^2$	$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$
$v_f^2 = v_i^2 + 2a\Delta x$	$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$
$\Delta x = (\frac{v_i + v_f}{2})t$	$\Delta\theta = (\frac{\omega_i + \omega_f}{2})t$

Ex 8.4 - A record player reaches its speed of 33 rpm after making 1.7 revolutions. What was its angular acceleration?

$\omega_i = 0 \text{ rad/sec}$

$\omega_f = 33\text{ rpm} \times \frac{2\pi \text{ rad}}{60 \text{ sec}} = 3.456 \text{ rad/sec}$

$\Delta\theta = 1.7\text{ rev} = 10.6814 \text{ rad}$

$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$

$3.456^2 = 0^2 + 2\alpha(10.6814)$

$\boxed{\alpha = .559 \text{ rad/sec}^2}$

Ex 8.5 – An automobile engine slows down from 4000 rpm to 1200 rpm in 3.5 s. Calculate (a) its angular acceleration, assumed uniform, and (b) the total number of revolutions the engine makes in this time.

$\omega_i = 4000\text{ rpm} \times \frac{2\pi \text{ rad}}{60 \text{ sec}} = 418.879 \text{ rad/s}$

$\omega_f = 1200\text{ rpm} \times \frac{2\pi \text{ rad}}{60 \text{ sec}} = 125.664 \text{ rad/s}$

$t = 3.5 \text{ sec}$

a. $\omega_f = \omega_i + \alpha t$

$125.664 = 418.879 + \alpha(3.5)$

$\boxed{\alpha = -83.77 \text{ rad/s}^2}$

b. $\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$

$= 418.879(3.5) + \frac{1}{2}(-83.77)(3.5)^2$

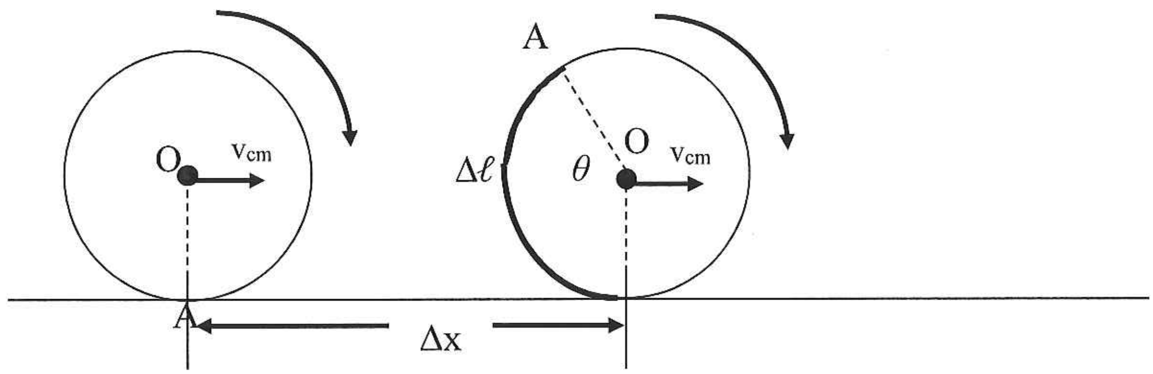
$\Delta\theta = 952.95 \text{ rad} \times \frac{1 \text{ rev}}{2\pi \text{ rad}}$

$\boxed{\Delta\theta = 151.67 \text{ revolutions}}$

Section 8.3 – Rolling Motion

Consider a ball or a wheel rolling across the floor. Rolling *without slipping* depends on static friction between the rolling object and the ground. It is static because the point of contact is *at rest* relative to the ground (Kinetic friction will come into play if you are sliding or skidding, such as on ice).

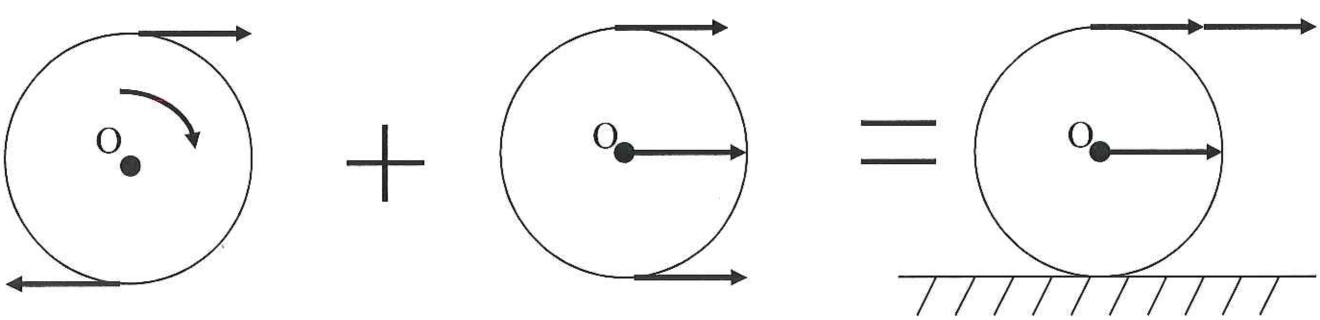
As the wheel rolls, a particular point (A) on the wheel travels a certain distance, which is the arc length of the circle shown below.



This is the same as the distance travelled (assuming no slippage). We know that $\Delta \ell = \theta\ r = \Delta x$. We also know that $v_{cm} = \dfrac{\Delta x}{\Delta t}$. Thus,

$$v_{cm} = \frac{\Delta x}{\Delta t} = \frac{\Delta(\theta r)}{\Delta t} = \frac{\Delta \theta}{\Delta t} r \qquad \text{and} \qquad \boxed{v_{cm} = \omega\ \mathrm{r}}$$

Rolling *without slipping* requires both rotation and translation.



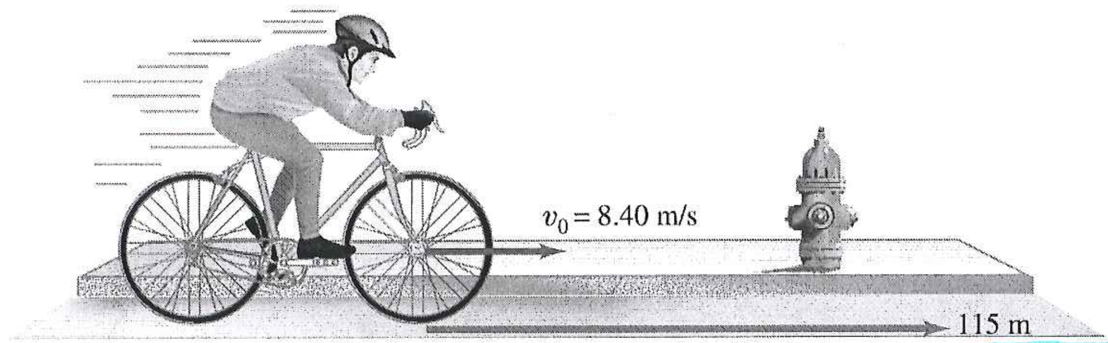
Rotation

Translation

Rotation and Translation

$\boxed{v_{top\ of\ wheel} = 2\,v_{cm}}$ $\boxed{v_{center\ of\ wheel} = v_{cm}}$ $\boxed{v_{bottom} = 0}$

Ex 8.6 - A bicycle slows down uniformly from v = 8.40 m/s to rest over a distance of 115 m. Each wheel and tire has an overall diameter of 68.0 cm. Determine (a) the angular velocity of the wheels initially; (b) the total number or revolutions each wheel rotates before coming to rest; (c) the angular acceleration of the wheel; and (d) the time it took to come to a stop.



Bike as seen from the ground at $t = 0$

$V_i = 8.4\,m/s$

$V_f = 0\,m/s$

$\Delta x = 115m$

$d = .68m$

$r = .34m$

a. $V = \omega r \rightarrow \omega_i = V/r = \frac{8.4\,m/s}{.34m} = \boxed{24.7\ rad/sec}$

b. $\Delta\theta = \frac{\Delta l}{r} = \frac{115m}{.34m} = 338.23\,rad \times \frac{1\,rev}{2\pi\,rad} = \boxed{53.83\,revs}$

c. $\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta \rightarrow 0^2 = 24.7^2 + 2\alpha(338.23)$

$\boxed{\alpha = -.901\,rad/s^2}$

d. $\omega_f = \omega_i + \alpha t \rightarrow 0 = 24.7 - .901t \rightarrow \boxed{t = 27.4\,sec}$

8–2 and 8–3 Problems

6. A cooling fan is turned off when it is running at 850 rev/min. It turns 1250 revolutions before it comes to a stop. (*a*) What was the fan's angular acceleration, assumed constant? (*b*) How long did it take the fan to come to a complete stop?

$\omega_i = \frac{850\,rev}{min} \times \frac{2\pi\,rad}{60\,sec} = 89.01\,rad/sec$

$\Delta\theta = 1250\,rev = 7,853.98\,rad$

$\omega_f = 0\,rad/sec$

a. $\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$

$0^2 = 89.01^2 + 2\alpha(7853.98)$

$\boxed{\alpha = -0.504\,rad/s^2}$

b. $\omega_f = \omega_i + \alpha t$

$0 = 89.01 + (-.504)t$

$\boxed{t = 176.6\,sec}$

7. A small rubber wheel is used to drive a large pottery wheel. The two wheels are mounted so that their circular edges touch. The small wheel has a radius of 2.0 cm and accelerates at the rate of 7.2 rad/s^2, and it is in contact with the pottery wheel (radius 27.0 cm) without slipping. Calculate (*a*) the angular acceleration of the pottery wheel, and (*b*) the time it takes the pottery wheel to reach its required speed of 65 rpm.

a. Since there is no slipping, the tangential displacement, velocity and acceleration must be the same:

$r_1 = .02m$

$\alpha_1 = 7.2 \, rad/s^2$

$r_1 = .25m$

$a_1 = a_2 \rightarrow \alpha_1 r_1 = \alpha_2 r_2 \rightarrow 7.2(.02) = \alpha_2(.25)$

$\boxed{\alpha_2 = .576 \, rad/s^2}$

b. $\omega_i = 0 \, rad/s$

$\omega_f = 65rpm = 6.807 rad/s$

$\alpha = .576 \, rad/s^2$

$\omega_f = \omega_i + \alpha t$

$6.807 = .576t$

$\boxed{t = 11.82 sec}$

8. A particle moves in a circle of radius 100 m with a constant velocity of 20 m/s. (a) What is the angular velocity in radians per second about the center of the circle? (b) How many revolutions does it make in 30 s?

$r = 100m$

$v = 20 m/s$

a. $\omega = v/r = 20 m/s / 100m = \boxed{.2 \, rad/sec}$

b. $30 \cancel{sec} \times \frac{.2 \, \cancel{rad}}{\cancel{sec}} \times \frac{1 rev}{2\pi \, \cancel{rad}} = \boxed{.955 rev}$

9. A 30 cm diameter music record (vinyl) rotates at 33 1/3 rev/min. (a) What is its angular velocity in radians per second? (b) Find the speed and <u>*linear*</u> acceleration of a point on the rim of the record. (c) If it then brakes with a constant angular acceleration and comes to rest in 2.0 min find the angular acceleration. (d) What is the average angular velocity of the turntable? (e) How many revolutions does it make before stopping?

a. $\omega = \frac{33 \, 1/3 \, rev}{min} \times \frac{1 min}{60 sec} \times \frac{2\pi \, rad}{1 rev} = \boxed{3.49 \, rad/sec}$

b. $v = \omega r = 3.49(.15) = \boxed{.5235 m/s}$

$a_{Tot} = a_{Rad} = \frac{v^2}{r} = \frac{.5235^2}{.15} = \boxed{1.82 m/s^2}$

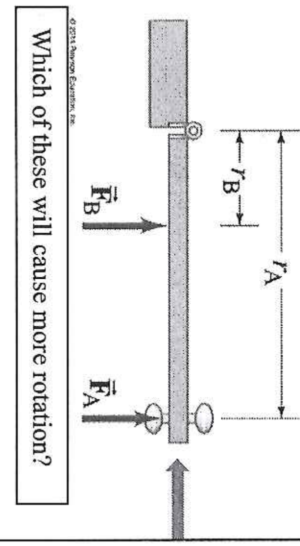
c. $\omega_f = \omega_i + \alpha t \rightarrow 0 = 3.49 + \alpha(120) \rightarrow \boxed{\alpha = -.029 \, rad/sec}$

e. $\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2 \rightarrow \Delta\theta = 3.49(120) + \frac{1}{2}(-.029)(120)^2 = 210 rad = \boxed{33.4 rev}$

d. $\bar{\omega} = (\omega_f + \omega_i)/2 = 3.49/2 = \boxed{1.745 \, rad/sec}$

Section 8.4 – Torque

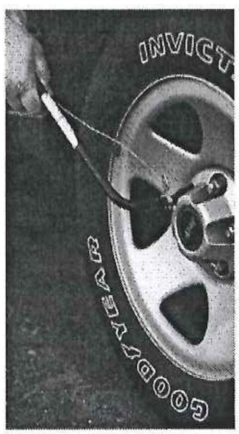
Think about a rotating object. Why is it rotating? What has caused this rotation? The answer is a concept we will call ***torque***. To make an object rotate about some axis a force is clearly required. But the *direction* that the force is applied certainly makes a difference.



This force won't cause any rotation!

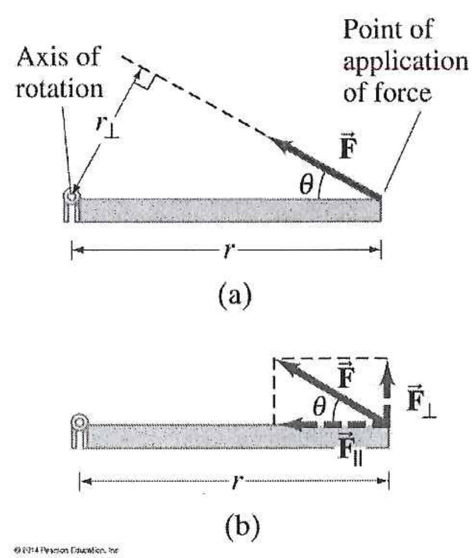
Imagine trying to open a door by pushing inward towards the hinge. You won't have much success getting the door to rotate on its axis regardless of how much force you apply! What direction will cause the door to open the easiest? Try it out and you'll find that pushing *perpendicular* on the door will cause the door to open the easiest.

But it also matters how far from the axis of rotation you apply the force. Which force, A or B on the diagram will cause more rotation? The further from the axis of rotation, the more ***torque*** the force produces! This is why tools like wrenches have such long handles.



This distance from the axis of rotation is called the ***lever arm*** or ***moment arm*** of the force, and we denote it with the variable *r*. The larger the value of *r* the more ***torque*** that is produced.

So we can define ***torque*** using the Greek letter tau τ as the *cross product* of *r* and *F*. Mathematically it is written as:

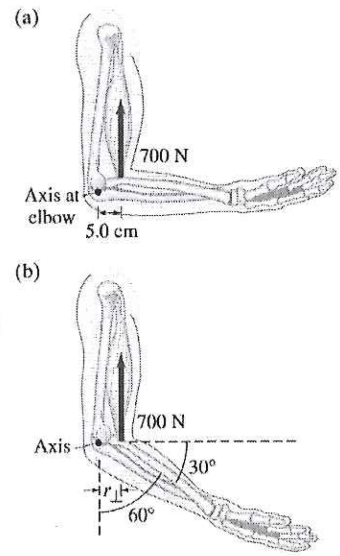


$$\tau = r \times F = rF\sin\theta$$

The sine of the angle comes from the cross product of two vectors and it represents the fact that 90 degrees (perpendicular) maximizes the amount of torque.

Units: The units of Torque are Newton meters (N m)

Ex 8.7 – The biceps muscle exerts a vertical force on the lower arm as shown. For each case, calculate the torque about the axis of rotation through the elbow joint, assuming the muscle is attached 5.0 cm from the elbow. (a) 700N of force; arm at 90 degree angle; (b) 700 N of force; arm at 120 degree angle.



a. $\tau = rF\sin\theta$

$= .05(700)\sin 90$

$\boxed{\tau = 35 Nm}$

b. $\tau = rF\sin\theta$

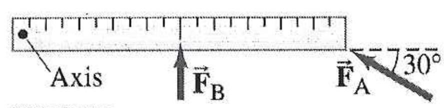
$= .05(700)\sin 120$

$\boxed{\tau = 30.31 Nm}$

Net Torque

When more than one torque acts on an object, the angular acceleration of the object is found by finding the ***net torque***, which is the sum of all torques acting on the object. When both torques act in the same direction, they add together. When they act in opposite directions, they subtract from one another. (Clockwise – negative; Counter-clockwise – positive)

Ex 8.8 – Two forces ($F_B = 20$ N and $F_A = 30$ N) are applied to a meter stick which can rotate about its left end. Force B is applied perpendicularly at the midpoint. Force A is applied at the end at the angle shown. (a) Which force exerts the greater torque? (b) What is the net torque?



a. $\tau_A = rF\sin\theta = 1(30)\sin 30$

$\tau_A = 15 Nm$

$\tau_B = .5(20)\sin 90 = 10 Nm$

$\boxed{F_A}$ exerts a greater torque.

b. $\Sigma\tau = \tau_A + \tau_B = 20 + 30 = \boxed{50 N \cdot m}$

8–4 Problems

10. Can a small force ever exert a greater torque than a larger force? Explain.

Yes, if the lever arm is greater.

11. A 52-kg person riding a bike puts all her weight on each pedal when climbing a hill. The pedals rotate in a circle of radius 17 cm. (*a*) What is the maximum torque she exerts? (*b*) How could she exert more torque?

$F_g = 52kg(9.8\,m/s^2) = 509.6N$

$r = .17m$

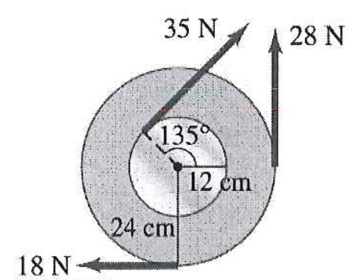
a. $\tau = rF\sin\theta = .17(509.6)\sin 90$

$\boxed{\tau = 86.632Nm}$

b. Push harder on the pedal

Pull up on the other pedal

12. Calculate the net torque about the axle of the wheel shown. Assume that a friction torque of 0.60 m • N opposes the motion.



$\tau_A = .12m(35N)\sin 90 = -4.2Nm$

$\tau_B = .24m(18N)\sin 90 = -4.32Nm$

$\tau_C = .24m(28N)\sin 90 = 6.72Nm$

$\tau_{fr} = .6N \cdot m$

To figure out motion: $-4.2 + (-4.32) + 6.72 = -1.8Nm$

So since its rotating clockwise, τ_{fr} is positive and

$\Sigma\tau = -1.8 + .6 \rightarrow \boxed{\Sigma\tau = -1.2Nm}$

13. A person exerts a horizontal force of 42 N on the end of a door 96 cm wide. What is the magnitude of the torque if the force is exerted (*a*) perpendicular to the door and (*b*) at a 60.0° angle to the face of the door?

$F = 42N$

$r = .96m$

a. $\tau = rF\sin\theta$

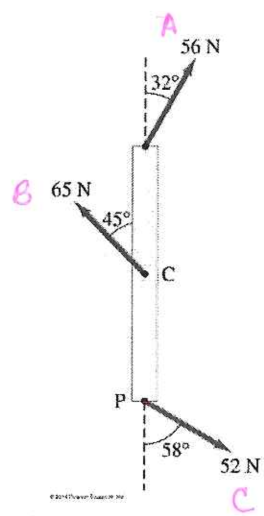
$= .96(42)\sin 90$

$\boxed{\tau = 40.32Nm}$

b. $\tau = rF\sin\theta$

$= .96(42)\sin 60$

$\boxed{\tau = 34.92Nm}$



14. Determine the net torque on the 2.0-m-long uniform beam shown. All forces are shown. Calculate about (*a*) point C, the CM, and (*b*) point P at one end.

-29.675

a. $\tau_A = 1m(56N)\sin 32 = \cancel{\quad} Nm$

$\tau_B = 0(65)\sin 45 = 0Nm$

$\tau_C = 1m(52N)\sin 58 = 44.099Nm$

$\Sigma\tau = -29.675 + 0 + 44.099 \rightarrow \boxed{\Sigma\tau = 14.42Nm}$

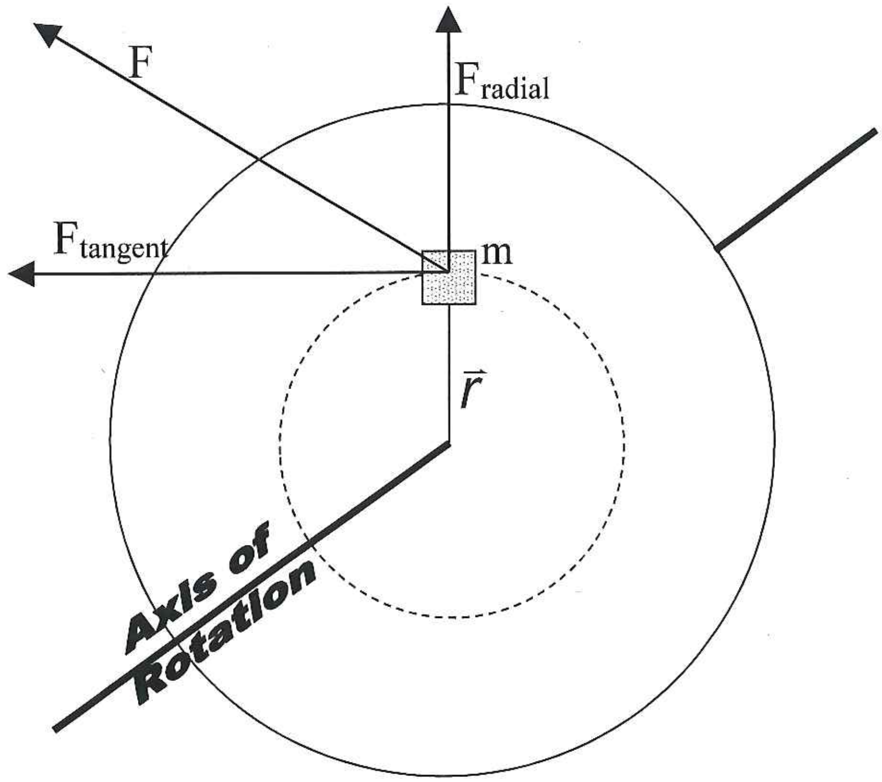
b. $\tau_A = 2m(56N)\sin 32 = -59.35Nm$

$\tau_B = 1m(65N)\sin 45 = 45.96Nm$

$\tau_C = 0Nm$

$\Sigma\tau = -59.35Nm + 45.96Nm \rightarrow \boxed{\Sigma\tau = -13.389Nm}$

8-5 Torque and Moment of Inertia



A force, F, is acting on a wheel F_{radial}, does not effect the rotation of the wheel. The torque exerted by F is

$\vec{\tau} = \vec{r} \otimes \vec{F} \quad \hat{n}$

BY DEFINITION

The magnitude of the torque, τ, "tau" is equal to the product of $F_{tangent}$, and the magnitude of the radius vector.

$$\boxed{\tau = F_{tan} r} \quad \text{(A)}$$

SINCE: $\boxed{F = ma}$ and $\boxed{a = \alpha r}$

$$\boxed{\tau = mr^2\alpha} \quad \text{(B)}$$

Now if we sum all of the particles, m, on the wheel.

$$\boxed{\Sigma\tau = \boxed{\Sigma mr^2}\alpha} \quad \text{(C)}$$

The sum, Σmr^2, is a property of the wheel called the moment of inertia, I. The moment of inertia depends on the distribution of the mass relative to the axis of rotation.

$$\boxed{I = \sum mr^2} \quad \text{(D)}$$

From Ⓒ and Ⓓ

$$\Sigma\tau = \Sigma mr^2\alpha \qquad I = \sum mr^2$$

$$\therefore \boxed{\Sigma\tau = I\alpha}$$

($\Sigma F = ma$)

Newton's 2nd Law for rotation

How to "think" about Torque

1. Torque must be specified about a pivot point
2. Torque is a product quantity made up of distance and force.
3. Torque causes angular acceleration, α, in the same way that forces cause linear accelerations.
4. The Moment of Inertia, I, is a measure of resistance to rotation analogous to mass as a measure of inertia for linear motion.
5. We can't "see" or "touch" torque, in the same sense that we can't "see" or "touch" forces – We see the effects of both.
6. Equilibrium means

$$\sum F = 0 \ \ and \ \ \sum \tau = 0$$

Object at rest totally
or translating and/or rotating with constant v or ω

8-6 Solving Problems in Rotational Dymanics

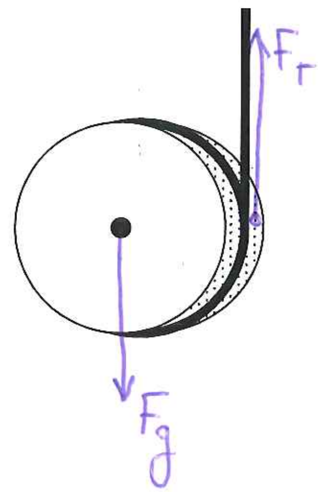
Example: A uniform cylinder of mass M and radius R is held by a hand that is accelerated upward so that the center of mass of the cylinder does not move. Find *(a)* the tension in the string, *(b)* the angular acceleration of the cylinder, and *(c)* the acceleration of the hand

Answers: (a) Mg (b) 2g/R (c) 2g

a. The center of mass does not move

So $\Sigma F = F_T - F_g = 0$

$\therefore \boxed{F_T = F_g = Mg}$



b. $\Sigma \tau = I\alpha$

F_g doesn't produce torque

F_T is only force providing torque

$\Sigma \tau = rF\sin\theta = I\alpha$

$rMg\sin 90 = I\alpha$ $\quad$ I for disk is $\frac{1}{2}Mr^2$

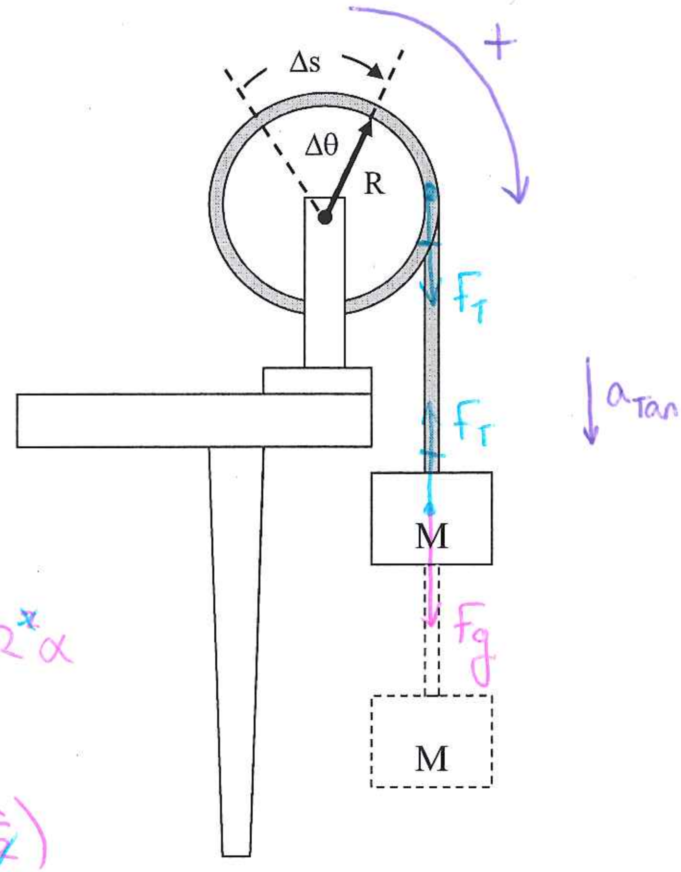
$\cancel{r}\cancel{M}g = \frac{1}{2}\cancel{M}r^{\cancel{2}}\alpha \rightarrow \boxed{\alpha = 2g/r}$

c. the acceleration of the hand must be the tangential acceleration of a point on the outmost point of cylinder.

Thus, $a = \alpha r$

$= \frac{2g}{r} \cdot r \rightarrow \boxed{a = 2g}$

Example: A body of mass 1.2 kg is tied to a light string wound around a 2.5 kg wheel of radius 0.2 m. The wheel bearing is frictionless. Find the **tension** is the string, the **acceleration** of the block, and its **speed** after it has fallen a distance h = 0.25 m from rest. *T=6N, a = -4.8 m/s², v = 1.55 m/s*



$\Sigma F = ma$

$-F_T + F_g = Ma$

$-F_T + Mg = Ma$

$\boxed{F_T} = Mg - Ma$

$\Sigma \tau = I\alpha$

$I = \frac{1}{2} m_p R^2$

$\Sigma \tau = \frac{1}{2} m_p R^2 \alpha$

$\cancel{R} F_T \cancel{\sin 90} = \frac{1}{2} m_p R^{\cancel{2}} \alpha$

$\alpha = a/R$

$F_T = \frac{1}{2} m_p \cancel{R} \left(\frac{a}{\cancel{R}}\right)$

$Mg - Ma_T = \frac{1}{2} m_p a_T \leftarrow \boxed{F_T} = \frac{1}{2} m_p a_{Tan}$

$Mg = \frac{1}{2} m_p a_T + Ma_T$

$1.2(9.8) = \left(\frac{1}{2}(2.5) + 1.2\right) a_T \rightarrow \boxed{a_T = 4.8\ m/s^2}$

$F_T = Mg - Ma$

$= 1.2(9.8) - 1.2(4.8) \rightarrow \boxed{F_T = 6N}$

$V_i = 0$

$\Delta x = .25m$

$a = 4.8\ m/s^2$

$V_f = ?$

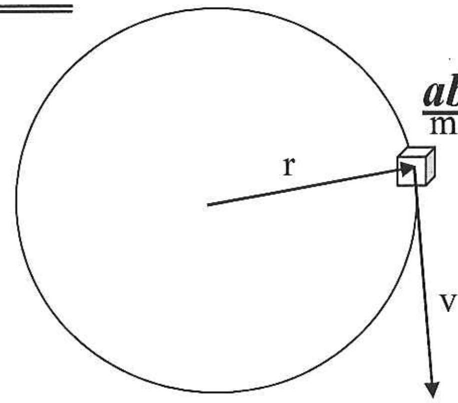
$V_f^2 = V_i^2 + 2a\Delta x$

$= 0^2 + 2(4.8)(.25)$

$\boxed{V_f = 1.55\ m/s}$

8-7 Rotational Kinetic Energy

Consider of piece of mass rotating about a central point



The Kinetic energy of the mass is

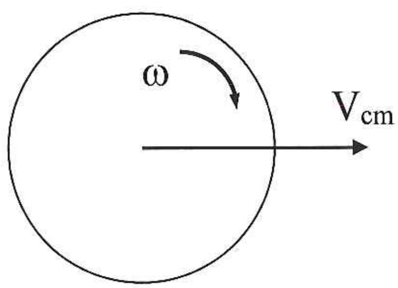
$$KE = \frac{1}{2} m\, v^2$$

Consider a wheel made up of many such masses, then

$$KE = \sum_i \frac{1}{2} m_i\, v_i^2 = \sum_i \frac{1}{2} m_i (r_i \omega)^2 = \frac{1}{2} (\sum_i m_i r_i^2) \omega^2$$

$$\boxed{KE_{rotation} = \frac{1}{2} I \omega^2}$$

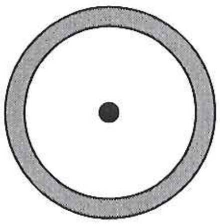
Remember that the wheel may also be translating



Then

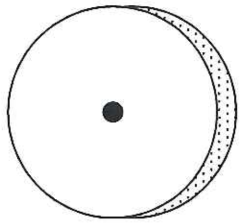
$$\boxed{KE_{total} = KE_{trans} + KE_{rotation} = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I \omega^2}$$

Rolling Without Slipping – Different Shapes
Moment of Inertia Considerations (Same Mass, Same Radius)



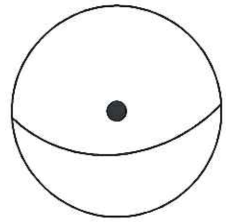
Hoop

$$I_{cm_{Hoop}} = M\,R^2$$



Disk

$$I_{cm_{Disk}} = \frac{1}{2} M\,R^2$$



Sphere

$$I_{cm_{Sphere}} = \frac{2}{5} MR^2$$

$$KE_{total} = KE_{rot} + KE_{trans}$$

$$KE_{total} = \frac{1}{2} I \frac{V_{cm}^2}{R^2} + \frac{1}{2} M\,V_{cm}^2$$

Hoop

$$\frac{1}{2}(MR^2)\frac{V_{cm}^2}{R^2} + \frac{1}{2}MV_{cm}^2$$

$$\underbrace{\frac{1}{2}MV_{cm}^2}_{\substack{50\ \% \\ KE_{rot}}} + \underbrace{\frac{1}{2}MV_{cm}^2}_{\substack{50\ \% \\ KE_{trans}}}$$

(mV_{cm}^2)

Disk

$$\frac{1}{2}(\tfrac{1}{2}MR^2)\frac{V_{cm}^2}{R^2} + \frac{1}{2}MV_{cm}^2$$

$$\underbrace{\frac{1}{4}MV_{cm}^2}_{\substack{33\ \% \\ KE_{rot}}} + \underbrace{\frac{1}{2}MV_{cm}^2}_{\substack{67\ \% \\ KE_{trans}}}$$

$(\frac{3}{4} mV_{cm}^2)$

Sphere

$$\frac{1}{2}(\tfrac{2}{5}MR^2)\frac{V_{cm}^2}{R^2} + \frac{1}{2}MV_{cm}^2$$

$$\underbrace{\frac{1}{5}MV_{cm}^2}_{\substack{29\ \% \\ KE_{rot}}} + \underbrace{\frac{1}{2}MV_{cm}^2}_{\substack{71\ \% \\ KE_{trans}}}$$

$(\frac{7}{10} mV_{cm}^2)$

If started from rest at the top of a hill, which will move the fastest at the bottom of a hill?

$$KE_i + PE_i = KE_f + PE_f$$

Hoop → Slowest!

$PE_i = KE_f$

$mgh = MV_{cm}^2$

$v = \sqrt{gh}$

Disk

$PE_i = KE_f$

$mgh = \frac{3}{4} MV_{cm}^2$

$v = \sqrt{\frac{4}{3} gh}$

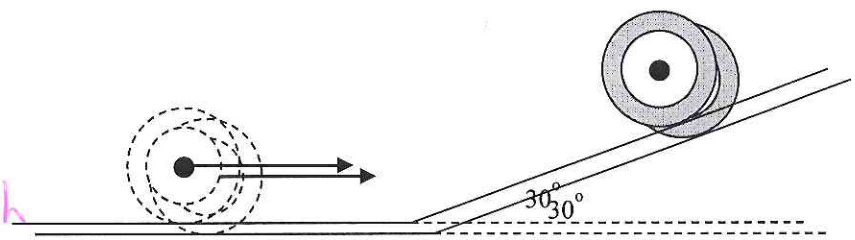
Sphere → fastest!

$PE_i = KE_f$

$mgh = \frac{7}{10} MV_{cm}^2$

$v = \sqrt{\frac{10}{7} gh}$

Example: A hoop of radius 0.5 m and mass 0.8 kg is rolling without slipping at a speed of 20 m/s toward an incline of slope 30°. How far along the incline will the hoop roll? Assuming it rolls with out slipping. *Answer 81.5 m*



$KE_R + KE_T = PE$

$\frac{1}{2} I \omega^2 + \frac{1}{2} m v^2 = mgh$

$\left(\frac{1}{2} \cancel{m} r^2\right) \omega^2 + \frac{1}{2} \cancel{m} v^2 = \cancel{m} gh$

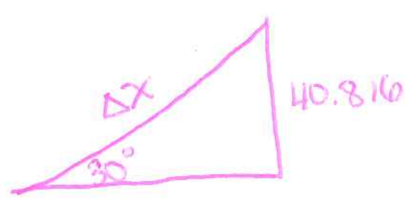
$\frac{1}{2} r^2 \left(\frac{v}{r}\right)^2 + \frac{1}{2} v^2 = gh$

$\frac{1}{2} v^2 + \frac{1}{2} v^2 = gh$

$v^2 = gh$

$20^2 = 9.8 h$

$h = 40.816 m$



$\sin 30 = 40.816 / \Delta x$

$\therefore \boxed{\Delta x = 81.6 m}$

Example: Four particles of mass m are connected by massless rods to form a rectangle of sides 2a and 2b, as shown below. The system rotates about an axis in the plane of the figure through the center. Find the moment of inertia about this axis and the kinetic energy of rotation if the angular velocity is ω

$I = \Sigma m r^2$

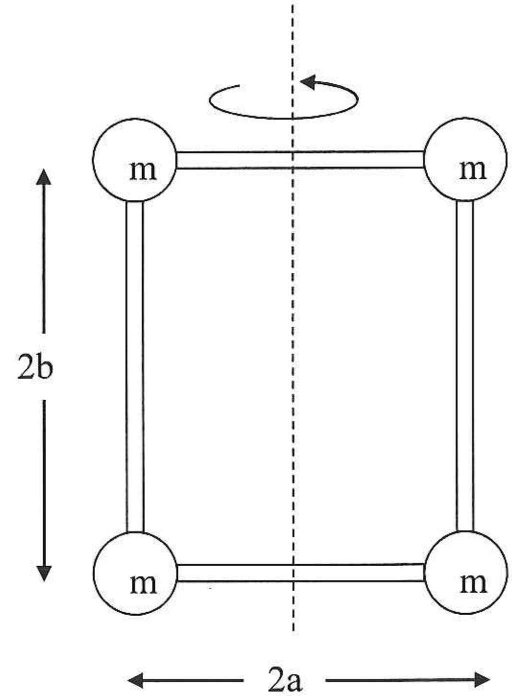
$= m(a)^2 + m(a)^2 + m(a)^2 + m(a)^2$

$\boxed{I = 4ma^2}$

$KE = \frac{1}{2} I \omega^2$

$= \frac{1}{2}(4ma^2)\omega^2$

$\boxed{KE = 2ma^2\omega^2}$



Example: Find the moment of inertia and Kinetic Energy of rotation of the system shown below.

$I = \Sigma m r^2$

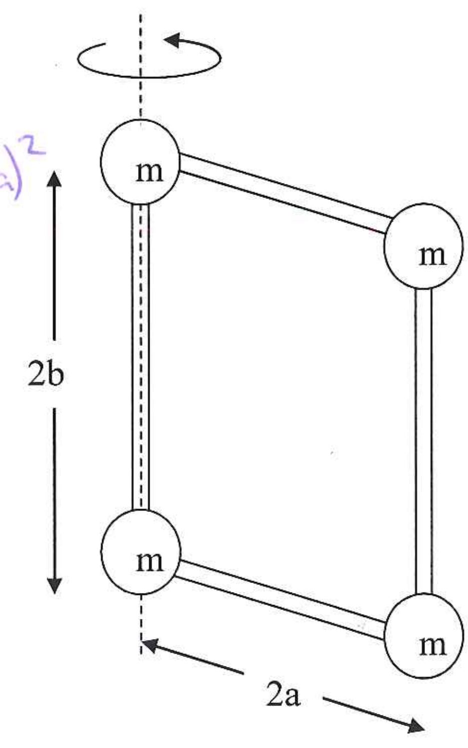
$= m(0^2) + m(0)^2 + m(2a)^2 + m(2a)^2$

$\boxed{I = 8ma^2}$

$KE = \frac{1}{2} I \omega^2$

$= \frac{1}{2}[8ma^2]\omega^2$

$\boxed{KE = 4ma^2\omega^2}$



8-8 Angular Momentum and Its Conservation

Definition of Angular Momentum

Recall

$$\sum F = m\,a$$

$$= m\frac{\Delta v}{\Delta t}$$

$$= \frac{\Delta(mv)}{\Delta t}$$

$$= \frac{\Delta p}{\Delta t}$$

with

$$p = mv$$

is Linear Momentum

Now

$$\sum \tau = I\,\alpha$$

$$= I\frac{\Delta \omega}{\Delta t}$$

$$= \frac{\Delta(I\omega)}{\Delta t}$$

$$\sum \tau = \frac{\Delta L}{\Delta t}$$

with

$$L = I\omega$$

is Angular Momentum

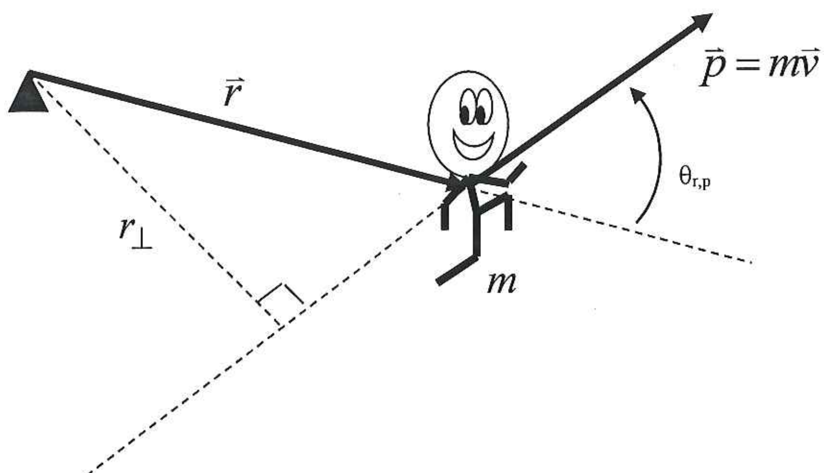
Since $$\sum \tau = \frac{dL}{dt}$$

If $\sum \tau = 0$ then $L_i = L_f$ and $I_i\omega_i = I_f\omega_f$

Torque and Angular Momentum – Formal Definitions

$\boxed{\vec{\tau} = \vec{r} \otimes \vec{F}}$ Torque analogous to Force

$\boxed{\vec{l} = \vec{r} \otimes \vec{p}}$ Angular momentum analogous to Linear Momentum



An object moving in a straight line has an angular momentum about the pivot point O.

We will do this when a particle moving in a straight line will be captured into rotational motion about that pivot point and $L_i = L_f$

before capture after capture

Newtons 2nd Law for Torques in momentum form

$$\vec{l} = \vec{r} \otimes \vec{p}$$

$$\tfrac{d}{dt}\vec{l} = \tfrac{d}{dt} m(\vec{r} \otimes \vec{v})$$

$$\frac{d\vec{l}}{dt} = m\left[(\vec{r} \otimes \tfrac{d\vec{v}}{dt}) + (\tfrac{d\vec{r}}{dt} \otimes \vec{v})\right]$$

$$\frac{d\vec{l}}{dt} = m\left[(\vec{r} \otimes \vec{a}) + (\vec{v} \otimes \vec{v})\right]$$

$$\frac{d\vec{l}}{dt} = (\vec{r} \otimes m\vec{a}) = \vec{r} \otimes \vec{F} = \vec{\tau}$$

$$\therefore \quad \boxed{\vec{\tau} = \frac{d\vec{l}}{dt}}$$

Conservation of Angular Momentum

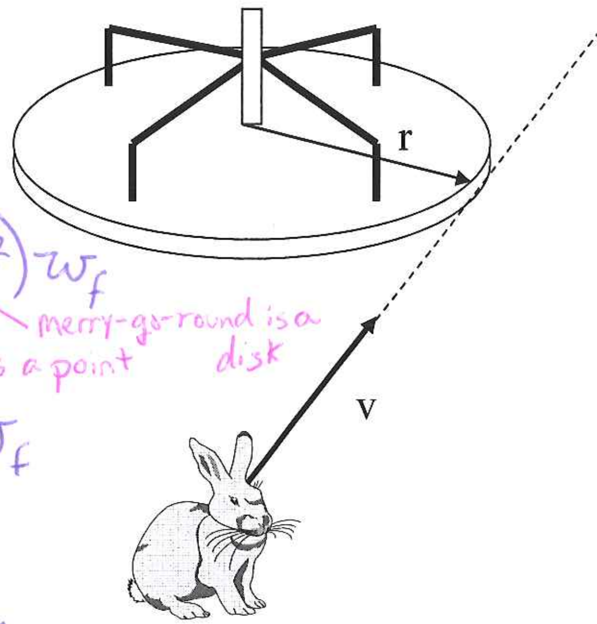
If the resultant external torque acting on a system is zero, the total angular momentum of the system is constant.

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad 0 = \frac{d\vec{L}}{dt} \qquad \therefore \qquad \vec{L} = \text{constant}$$

only if there are no external torques

Conservation of Angular Momentum

Example: A playground merry-go-round is at rest, pivoted about a frictionless axis. A ferocious rabbit runs along a path tangential to the rim with initial speed v_i and jumps onto the merry-go-round. What is the angular velocity of the merry-go-round and the rabbit?



$$L_i = L_f$$

$$I_r \omega_r + I_m \omega_m = I_{m+r} \omega_{m+r}$$

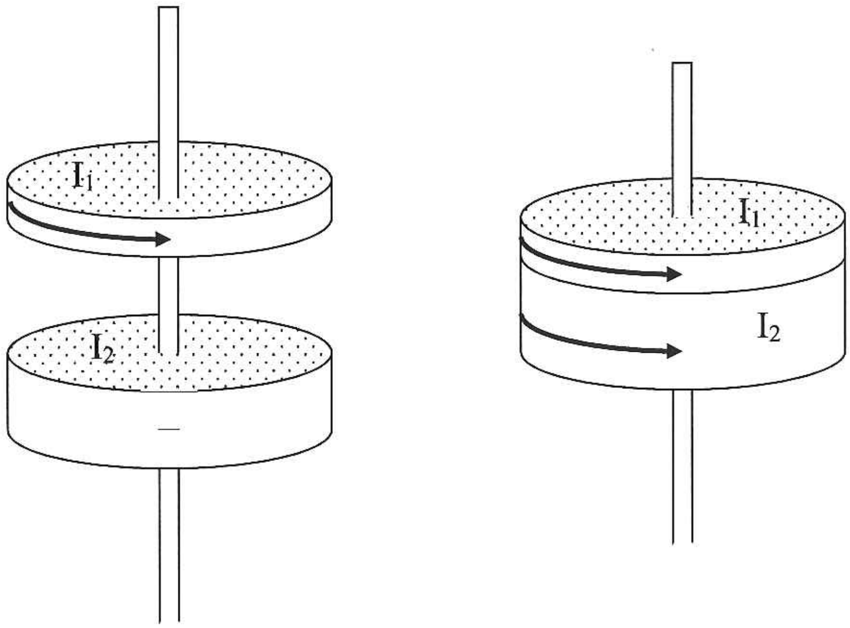
$$(m_R r^2)\left(\frac{v_r}{r}\right) + 0 = \left(m_R r^2 + \frac{1}{2} M r^2\right)\omega_f$$

rabbit is a point — merry-go-round is a disk

$$m_R r v_R = \left(m_R r^2 + \frac{1}{2} M r^2\right)\omega_f$$

Typically you are given all the quantities and asked to solve for ω_f.

Example: A disk with a moment of inertia I_1 is rotating with angular velocity ω_1 about a frictionless shaft. It drops onto another disk with moment of inertia I_2 initially at rest. Because of surface friction, the two disks eventually attain a common angular velocity. What is it?



$$L_i = L_f$$

$$I_1 \omega_i + \cancel{I_2 \omega_2}^{\,0} = (I_1 + I_2)\,\omega_f$$

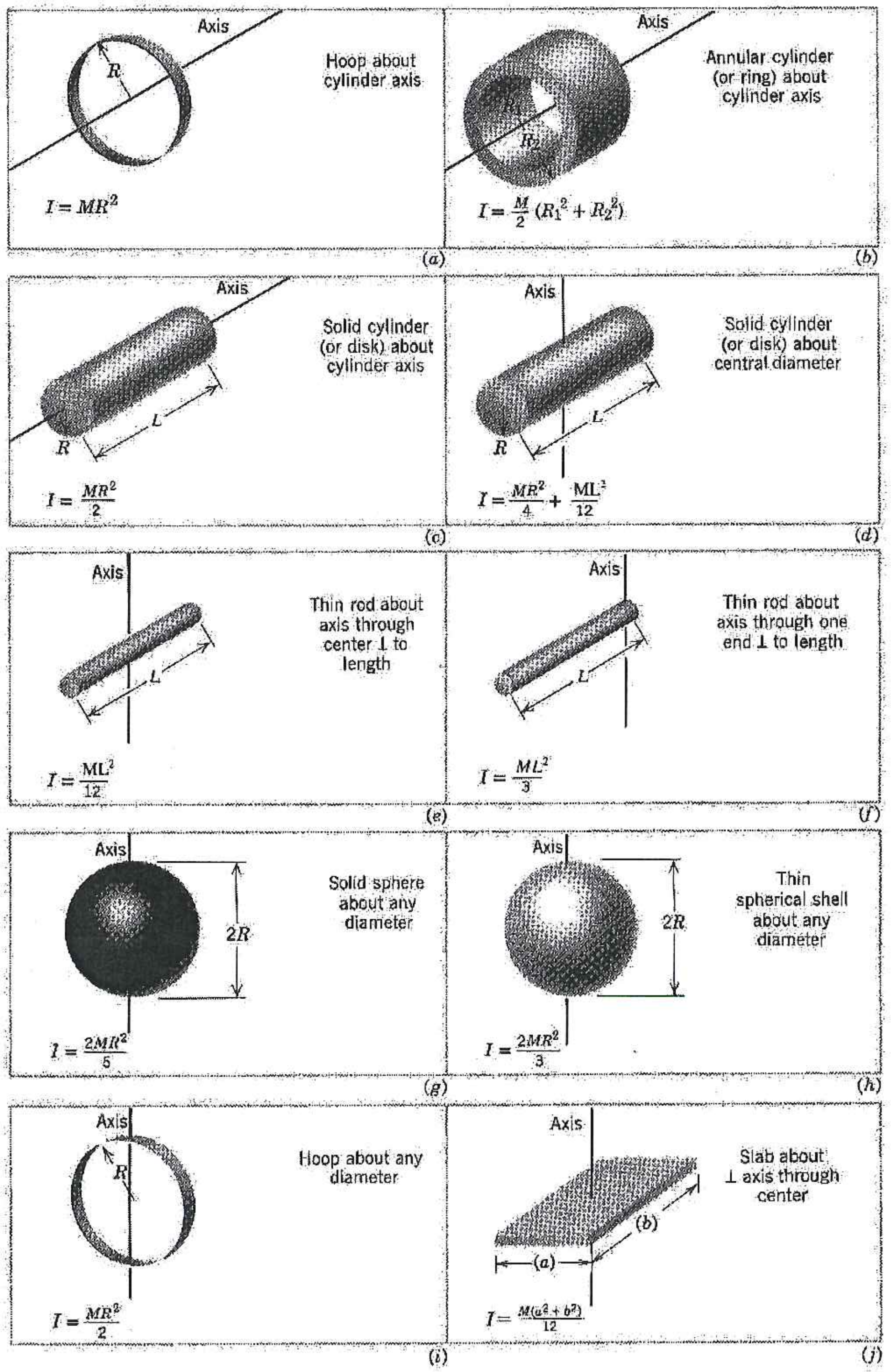
$$\boxed{\omega_f = \left(\frac{I_1}{I_1 + I_2}\right)\omega_i}$$

if you want to simplify more:

$$\omega_f = \left(\frac{\cancel{\tfrac{1}{2}} m_1 \cancel{r^2}}{\cancel{\tfrac{1}{2}} m_1 \cancel{r^2} + \cancel{\tfrac{1}{2}} m_2 \cancel{r^2}}\right)\omega_i$$

$$\therefore\ \omega_f = \left(\frac{m_1}{m_1 + m_2}\right)\omega_i$$

Appendix II - Some Rotational Inertias



Comparison of Linear Motion and Rotational Motion

Linear Motion		Rotational Motion	
Displacement	Δx	Angular Displacement	$\Delta\theta$
Velocity	$v = \frac{dx}{dt}$	Angular Velocity	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	Angular Acceleration	$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
Constant Acceleration Equations	$x = x_o + v_o t + \frac{1}{2}at^2$	Constant Angular Acceleration Equations	$\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$
	$v = v_o + at$		$\omega = \omega_o + \alpha t$
	$v^2 = v_o^2 + 2a\Delta x$		$\omega^2 = \omega_o^2 + 2\alpha\Delta\theta$
	$v_{av} = \frac{1}{2}(v_o + v)$		$\omega_{av} = \frac{1}{2}(\omega_o + \omega)$
Mass	m	Moment of Inertia	I
Momentum	$\vec{p} = m\vec{v}$	Angular Momentum	$\vec{L} = I\vec{\omega}$
Force	$\vec{F}$	Torque	$\vec{\tau}$
Power	$P = \vec{F} \bullet \vec{v}$	Power	$P = \vec{\tau} \bullet \vec{\omega}$
Newton's 2nd Law	$\sum F = \frac{d\vec{p}}{dt} = m\vec{a}$	Newton's 2nd Law	$\sum \tau = \frac{d\vec{L}}{dt} = I\vec{\alpha}$